Matrices

For each of the following matrices, Determine (true/false) whether each of the following matrices is: invertible, real symmetric, positive definite, orthogonal.

Definition: A is invertible if A^{-1} exists.
Definition: A is real symmetric if it is a real matrix and $A' = A$.
Definition: A is orthogonal if it is a real matrix and $A' = A^{-1}$.
Definition: A is positive definite if it is real symmetric and all of its eigenvalues are positive real values.
Definition: A is normal if $AA' = A' A$.

matrix	invertible?	real symmetric?	orthogonal?	positive definite?
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	**True** ☐ **False** ☐	**True** ☐ **False** ☐	**True** ☐ **False** ☐	**True** ☐ **False** ☐
$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	**True** ☐ **False** ☐	**True** ☐ **False** ☐	**True** ☐ **False** ☐	**True** ☐ **False** ☐
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	**True** ☐ **False** ☐	**True** ☐ **False** ☐	**True** ☐ **False** ☐	**True** ☐ **False** ☐
$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$	**True** ☐ **False** ☐	**True** ☐ **False** ☐	**True** ☐ **False** ☐	**True** ☐ **False** ☐

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Definition: A is normal if $AA' = A' A$.

For each of the following equations, mark whether the equation is True (valid for all specified matrices A) or False. Assume that $A^\top$ denotes the transpose of A, A' denotes the hermitian transpose (conjugate transpose) of A, and $i = \sqrt{-1}$.

Assume for real θ that 2×2 rotation matrices are $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$, and reflections are $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$.

True ☐ **False** ☐ Every product of two 2×2 rotation matrices is orthogonal.

True ☐ **False** ☐ If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the eigenvalues of A are $\lambda = \left((a+d) \pm \sqrt{(a+d)^2 - 4(a\,d - b\,c)}\right)/2$.

True ☐ **False** ☐ If $A = Q\ L\ Q'$, where L is real diagonal and Q orthogonal, then A is symmetric.

True ☐ **False** ☐ CS170A is the best course at UCLA.

True ☐ **False** ☐ If the eigenvalues of a square matrix A are all real and positive, then A is invertible.

True ☐ **False** ☐ If a matrix is real symmetric and also orthogonal, then it is the identity matrix.

True ☐ **False** ☐ For all real θ and ϕ, the product $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$ is a rotation.

True ☐ **False** ☐ If $A = Q\ L\ Q'$, where L is diagonal, and Q is orthogonal, then $A' = Q'\ L\ Q$.

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Assume for real θ that 2×2 rotation matrices are $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$, and reflections are $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$.

True ☐ **False** ☐ If A, is a symmetric real matrix, then $B = A'\,A$ is positive definite.

True ☐ **False** ☐ If $A = Q\,L\,Q^\top$, where L is a diagonal real matrix whose diagonal entries are negative, and Q is orthogonal, then A^2 is positive definite.

True ☐ **False** ☐ If A has the eigenvalue decomposition $A = Q\,L\,Q'$, where L is the diagonal matrix of eigenvalues of A and Q is a real orthogonal matrix, and A is invertible, then A is positive definite.

True ☐ **False** ☐ If $\det(A)$ is a positive real number, then A is positive definite.

Characteristic Polynomials

Give an example of a 2×2 normal matrix A whose characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ has roots ± 2.

$$A = \begin{pmatrix} \quad & \quad \end{pmatrix}$$

What are the eigenvalues and eigenvectors of your matrix? (Hint: trace(A) is the sum of eigenvalues, det(A) is the product.)

$$\lambda_1 = \qquad \lambda_2 = \qquad \mathbf{e}_1 = \begin{pmatrix} \quad \end{pmatrix} \qquad \mathbf{e}_2 = \begin{pmatrix} \quad \end{pmatrix}$$

MATLAB

Consider the following MATLAB statements:

```
n = 3
i = (1:n)' * ones(1,n)
j = i'
A = ~ (abs(i - j) > 1)    %   "~" is the MATLAB negation operator
```

What matrix `A` does execution of these statements produce?

☐ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Consider the following MATLAB functions:

```
L2_norm @(A) sqrt(max(eig(A * A')));  % square root( largest eigenvalue of A A' )
L1_norm = @(A) max(sum(abs(A)))       % largest col sum of |A|  (= abs(A)).
F_norm = @(A) sqrt(sum(sum(A.^2))))   % square root of the sum of squares of all elements in A.
definite = @(A) logical(prod(double(eig(A)>0)))  % all eigenvalues of A are positive
```

Fill in the following table with the values returned by these functions for the following matrices `A`:

matrix `A`	`L2_norm(A)`	`L1_norm(A)`	`F_norm(A)`	`definite(A)`
$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$				
$\begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$				
$\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$				
$\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$				