# Session 8

## Example of quadratic form and its extrema

Wednesday, Nov 17, 2021

Suppose $Q(\bar{x}) = Q(x,y) = x^2 - 2xy + y^2$ for $\bar{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

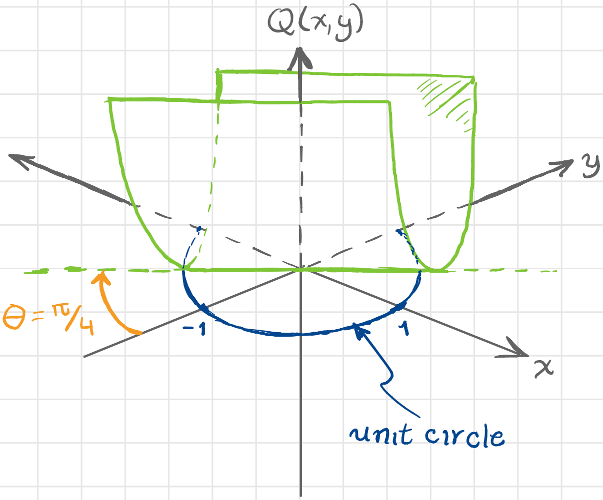
We can express $Q(x,y)$ in matrix form $\bar{x}^T A \bar{x} =: Q(\bar{x})$, where

$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ because $\underbrace{\begin{bmatrix} x & y \end{bmatrix}}_{\bar{x}^T} \underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\bar{x}} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} x-y \\ y-x \end{bmatrix}$

$= x^2 - xy + y^2 - xy$

$= x^2 - 2xy + y^2$

The plot for this quadratic form is:




and, we'd like to find the extrema of $Q(\bar{x})$, subject to the constraint:

$$x^2 + y^2 = 1$$

$$\Updownarrow$$

$$\|\bar{x}\|_2^2 = 1$$

Solution: According to the relation between $Q(x,y)$ and the eigenvector problem, the critical points are the eigenvectors of A, and the critical values are the corresponding eigenvalues.

Therefore, we compute $(\lambda, \bar{\omega})$ for A:

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow \lambda_0 = 0 \text{ for } \bar{\omega}_0 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}, \text{ and}$$

$$\lambda_1 = 2 \text{ for } \bar{\omega}_1 = \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$$

And, the minimum of $Q(x,y) = x^2 - 2xy + y^2$ is $0 =: \lambda_0$ attained when $x = \sqrt{2}/2$ and $y = \sqrt{2}/2$; and the maximum is $2 =: \lambda_1$ attained when $x = \sqrt{2}/2$ and $y = -\sqrt{2}/2$